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The existence and uniqueness of small-angle scattering solutions to equations of particle motion with radiation reaction

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Abstract. We study a general type of equation of motion for two point particles in flat space–time, involving radiation reaction terms and previous states of motion of the particles at retarded and multiply retarded times: the type includes the Lorentz–Dirac electromagnetic equation and approximate gravitational equations, and may be applicable to approximations to other nonlinear theories. Existence and uniqueness are proved subject to given asymptotic conditions in the infinite past and regularity conditions at all times, provided the coupling constant is sufficiently small (i.e. small-angle scattering).

1. Introduction

In recent years, radiation reaction calculations both in general relativity (Ehlers *et al* 1975, Rosenblum 1978, 1981, Kates 1980) and in gauge theories of strongly interacting particles (Drechsler 1980, Trautman 1981) have become very important. In general relativity, theoretical calculations of the period change of a binary system due to radiation reaction can be compared with Taylor's observations of the binary pulsar (PSR 1913 + 16) to provide an excellent test of the theory. In strong interaction physics, the results of radiation reaction calculations for classical SU(3) Yang–Mills fields may be of help for the problem of quark confinement (Drechsler and Rosenblum 1981).

Since both Einstein's equation and the Yang–Mills equation are nonlinear, approximation methods must be used to obtain approximate solutions which can be related to the sources. In Lorentz-covariant approximation methods, space and time are kept on the same footing in the various iterations of the nonlinear field equation. This approximation method is obviously very good for studying radiation problems. When the nonlinear field equations for a two body problem are solved approximately and the resulting fields are placed in the law of motion, it is found that the equations of motion for a particle are not only functionals of the other particle's mechanical variables evaluated at the retarded time, but can in addition be functionals of the particle's own variables evaluated at the retarded time of the retarded time (Rosenblum 1981, Bertotti and Plebanski 1960). The effect follows from the iteration of a nonlinear field equation.

Since the resulting force law in general relativity has been solved approximately in the case of small-angle scattering (Rosenblum 1978), it is of great importance to know whether or not the approximate force law which contains multiply retarded terms has solutions which exist and are unique.

To this end we consider two particles moving in Minkowski space–time under the influence of retarded or multiply retarded interactions with radiation reaction terms. Existence and uniqueness theorems are proved in outline for small-angle scattering with initial conditions prescribed in the infinite past and a condition in the infinite future to preclude runaway solutions when the formal damping term is positive.

The next section, 2, describes the equations to be considered with their associated boundary conditions, after which the main theorem is stated. In § 3 we briefly review the ideas of the proof. This is based in part on the techniques of Flume (1971), who proved existence for the symmetrical electrodynamic interaction. Finally, in § 4 we outline the proof, omitting purely routine calculations that exactly parallel parts of Flume's work.

2. Specification of the problem

An important feature of the equations of motion to be considered is the occurrence of multiply retarded times. If $x_1(\tau)=(x_1(\tau)^0,\ldots,x_1(\tau)^3)$ defines the world line of particle 1 as a function of parameter τ, and similarly for $x_2(\tau)$ in the case of particle 2, then these world lines determine a retardation function R_{12} by the condition that $x_1(\tau)-x_2(R_{12}(\tau))$ is null and future pointing: $R_{12}(\tau)$ is the retarded parameter for particle 2 corresponding to parameter τ for particle 1. R_{21} is defined similarly. By composition we can define multiple retardation functions $R_{21}\circ R_{12}$, $R_{12}\circ R_{21}\circ R_{12}$ and so on. The general equations of motion to be considered are of the form

$$\ddot{x}_1(\tau)=t_1[\dddot{x}_1(\tau)+f_1(x_1(\tau),x_2(R_{12}(\tau)),x_1(R_{21}\circ R_{12}(\tau)),\ldots,\dot{x}_1(\tau),$$
$$\dot{x}_2(R_{12}(\tau)),\ldots,\ddot{x}_1(\tau),\ddot{x}_2(R_{12}(\tau)),\ldots)] \tag{1}$$

together with an equation for $\ddot{x}_2$ derived by interchanging suffixes 1 and 2. Multiple retardations may be involved for all the variables indicated, up to a level of N retardations. The t_A are constants with the dimensions of length (e.g. $t_A=e^2/m_A$ for electromagnetism or Gm_A for gravitation). The velocity of light, c, is 1 and time is measured in length units. For electromagnetism t_A, which plays the role of a radiative damping time scale, is positive, while for gravitation it is negative. (This does not mean that gravitational motion is anti-damped, because this term is cancelled out by higher-order terms in the fast-motion procedure (Smith and Havas 1965), as must occur because of the absence of gravitational dipole radiation.) We shall describe the case $t_A>0$, indicating modifications necessary for $t_A<0$.

From the point of view of existence and uniqueness theorems, equations involving multiple retardations are no more difficult to handle than those involving single retardations, provided that the dependence of f on the velocities at multiply retarded times is of a particular form. More specifically, we require

$$f_A=f_A^0+f_A^* \qquad (A=1,2)$$

where f_A^* falls off rapidly with increasing separation of the particles (as specified by (4) below) and f_A^0 has the same form as the leading terms in the electromagnetic interaction (the 'Coulomb part'), namely

$$f_1^0=(2Z_1/\rho_1(\tau)^2)\dot{x}_2(R_{12}(x))^{[\mu}u_1(\tau)^{\nu]}\dot{x}_1(\tau) \tag{2}$$

(and similarly with 1 and 2 interchanged) where the Z_A are constants (determined by the charges, in electromagnetism) and

$$\rho_1(\tau) = -\dot{x}_2(R_{12}(\tau))_\mu(x_2(R_{12}(\tau)) - x_1(\tau))^\mu,$$
$$u_1(\tau)^\nu = \rho_1(\tau)^{-1}(x_2(R_{12}(\tau)) - x_1(\tau))^\nu - \dot{x}_2(R_{12}(\tau))^\nu$$

(and with 1 and 2 interchanged).

We can use f_A^0 to write down an approximation for the motion in the remote past. First, apply a Lorentz transformation so that the asymptotic initial four-velocities of the particles are

$$\left.\begin{aligned} v_1 &= (v^0, v^1, 0, 0) \\ v_2 &= (v^0, -v^1, 0, 0) \end{aligned}\right\} \qquad (\tau \to -\infty).$$

A 'zero-order' solution is then $x_A(\tau) = b_A + \tau v_A$. If we insert this in the right-hand side of the equations of motion with f_A^* omitted, and discard lower-order terms, we derive the functions

$$\xi_A(\tau) := \begin{cases} b_A + \tau v_A + \dfrac{Z_A t_A \log(-\tau)}{4(v^0 v^1)^2} v_A^* & \tau < -1 \\ b_A + \tau v_A & \tau \geqslant -1 \end{cases} \tag{3}$$

where $v_1^* = (v^1, -v^0, 0, 0)$, $v_2^* = (-v^1, -v^0, 0, 0)$.

We shall show the existence of solutions that approximate ξ_A in the sense of (iii) in the following set of boundary conditions, which we impose on the problem throughout.

(i) Each $x_A(\tau)$ is defined for all $-\infty < \tau < \infty$.

(ii) For some $K > 0$ and $\gamma > 0$, $|\ddot{x}_A(\tau)| < K(1+\tau)^{-\gamma}$ for all τ (where $|\ |$ is the Euclidean norm).

(iii) For some $K' > 0$ and α, $0 < \alpha < 1$, $|\ddot{x}_A(\tau) - \xi_A(\tau)| < K(1-\tau)^{-(2+\alpha)}$ for all $\tau < 0$.

(iv) For some $K'' > 0$, $|\rho_A(\tau)| > K''$ for all τ, where $\rho_A(\tau)$ (defined above) is the retarded spatial separation of the particles in the rest frame of A at proper time τ.

The motivation for conditions (ii)–(iv) is as follows. Condition (iii) formulates the asymptotic initial conditions, in terms of the approach of the trajectories to functions ξ with specified velocities and impact parameter in the infinite past. Condition (iv) bounds the separation of the particles from below and thereby restricts attention to small-angle scattering. Condition (ii) seems to be required, in addition to (iv), in order to ensure that the final state is 'free' (a genuine scattering), and hence to define in a suitable way the convergence of an iteration scheme throughout the whole of the trajectories. For $t_A > 0$, (ii) can be viewed as a means of excluding 'run-away' solutions. Note that Flume (1971) uses conditions at a finite time rather than asymptotic conditions.

In order to state the theorem, we introduce the following notation for the arguments of the functions f_A in equation (1). Write

$$\overset{(0)}{x}_A \equiv x_A, \qquad \overset{(1)}{x}_A \equiv \dot{x}_A, \qquad \overset{(2)}{x}_A \equiv \ddot{x}_A,$$

$$\overset{(p)}{x}_{10} \equiv \overset{(p)}{x}_1, \qquad \overset{(p)}{x}_{11} \equiv \overset{(p)}{x}_2 \circ R_{12}, \qquad \overset{(p)}{x}_{12} \equiv \overset{(p)}{x}_1 \circ R_{21} \circ R_{12}, \qquad \text{etc},$$

$$\overset{(p)}{x}_{20} \equiv \overset{(p)}{x}_2, \qquad \overset{(p)}{x}_{21} \equiv \overset{(p)}{x}_1 \circ R_{21}, \qquad \overset{(p)}{x}_{22} \equiv \overset{(p)}{x}_2 \circ R_{12} \circ R_{21}, \qquad \text{etc}.$$

Set also

$$\mu = \mu(\tau) \equiv \min(\rho_1(\tau), \rho_2 \circ R_{12}(\tau), \rho_1 \circ R_{21} \circ R_{12}(\tau) \text{ etc},$$
$$\rho_2(\tau), \rho_1 \circ R_{21}(\tau), \rho_2 \circ R_{12} \circ R_{21}, \text{etc}),$$
$$a \equiv a(\tau) = \max_{Aq} |\overset{(2)}{x}_{Aq}|, \qquad \kappa \equiv \kappa(\tau) = \max_{Aq} |\overset{(1)}{x}_{Aq}|.$$

We write also $t_A = M_A \varepsilon$ (thinking of ε as a universal constant) so as to be able to let $\varepsilon \to 0$ in the statement of the theorem.

Theorem. Suppose $f_A = f^0_A + f^*_A$, with f^0_A given by (2), satisfies

$$|f^*_A| \leqslant K[(a\mu^{-1} + \mu^{-2}) + |\overset{(2)}{x}_{A0}|^2], \tag{4a}$$

$$\left| \frac{\partial f^*_A}{\partial \overset{(p)}{x}{}^{\nu}_{Aq}} \right| \leqslant K^{(p)}_2 (a\mu^{p-2} + \mu^{p-3}) \qquad (p = 0, 1, 2; \nu = 0, \dots, 3; q = 0, 1, \dots, N) \tag{4b}$$

for some K_1, $K^{(p)}_2$, functions of κ. Then, for any constants K, K', K'', if ε is sufficiently small there exists a unique solution to (1) satisfying the boundary conditions (i)–(iv) above.

Remarks

(1) The conditions are all satisfied for electrodynamics, and on physical grounds are likely to be satisfied for most other reasonable interactions.

(2) In practice ε will be a fixed coupling constant. But the condition 'ε small' is equivalent (on dimensional grounds) to 'b_A and K (the impact parameters) large', i.e. to small-angle scattering.

(3) Note finally that we are not assuming that τ is proper time. In the case of electrodynamics τ can be shown to be proper time for any solution of (1) satisfying the boundary conditions with v_A normalised vectors.

3. Ideas of the proof

We take as our basic variables not the x_A but the accelerations $y_A = \ddot{x}_A$. Then x_A is reconstructed from $\ddot{x}_A$ by the identity

$$x_A(\tau) = \xi_A(\tau) + \int_{-\infty}^{\tau} \int_{-\infty}^{\tau'} (y_A(\tau'') - \ddot{\xi}_A(\tau''))\, d\tau''\, d\tau' \tag{5}$$

(which depends on the validity of asymptotic condition (iii)).

As in most treatments of the problem, we use condition (ii) to rewrite the equations as

$$\ddot{x}_A(\tau) = \int_{\tau}^{\infty} \exp[(\tau - \tau')/t_A] f_A(x_A(\tau'))\, dt'. \tag{6}$$

(For $t_A < 0$ the integral is $\int_{-\infty}^{\tau}$.) Then the proof is essentially a formalisation of the obvious iteration scheme based on (5) and (6). One starts with a first approximation $\ddot{x}^{(1)}_A = \ddot{\xi}_A$, then computes $x^{(1)}_A$ using (5) and inserts these in the right-hand side of (6).

Let us call the value of the functional defined by this process $\Gamma_A[\ddot{x}_1^{(1)}, \ddot{x}_2^{(1)}] = \ddot{x}_A^{(2)}$. Thus the iteration scheme is $\ddot{x}_A^{(n)} = \Gamma_A[\ddot{x}_1^{(n-1)}, \ddot{x}_2^{(n-1)}]$. Equation (6), in this notation, is

$$\ddot{x}_A = \Gamma_A[\ddot{x}_1, \ddot{x}_2]$$

i.e. $\ddot{\boldsymbol{x}} = (\ddot{x}_1, \ddot{x}_2)$ is a fixed point of the mapping $\boldsymbol{\Gamma}$: $\ddot{\boldsymbol{x}} \mapsto (\Gamma_1(\ddot{\boldsymbol{x}}), \Gamma_2(\ddot{\boldsymbol{x}}))$. Such a point will exist if the sequence $\ddot{\boldsymbol{x}}^{(n)}$ converges (with respect to a suitable topology on the space of all $\ddot{\boldsymbol{x}}$ satisfying asymptotic conditions (i)–(iv)) to a point in the domain of $\boldsymbol{\Gamma}$.

In Flume's treatment this space is given a compact-open topology by using a set of half-norms defined on a compact subspace of the proper-time-parameter space $\mathbb{R}$. In effect, he calculates the result of applying $\boldsymbol{\Gamma}$ in terms of these half-norms and then applies the Tychonoff fixed-point theorem to deduce the existence of a fixed point for $\boldsymbol{\Gamma}$. The above iteration is not explicitly used, but it is implicitly present through the proof of the Tychonoff theorem. We take over, more or less, his estimates for the effect of Γ (extending them to non-symmetric interactions and multiple retardations), but apply them in the context of a topology on a function space having the metric

$$d(\boldsymbol{y}, \boldsymbol{y}') = \|y_1 - y'\| + \|y_2 - y_2'\|$$

where $\|\ \|$, a norm adapted to the asymptotic conditions, is

$$\|y\| := \sup_{\tau<0}(1-\tau)^{2+\alpha}|y(\tau)| + \sup_{\tau>0}(1+\tau)^2|y(\tau)| \tag{7}$$

for $\mathbb{R}^4$-valued functions y on $\mathbb{R}$ having the norm defined, $|\cdot|$ being the Euclidean norm on $\mathbb{R}^4$.

The basis of the proof, which shows both that the iteration scheme converges to a fixed point and that this point is unique, is the demonstration that $\boldsymbol{\Gamma}$ is a contraction mapping, for small enough ε, provided it is restricted to the set

$$X_{s,\lambda,r} := \{\boldsymbol{y}: \Lambda_s(\boldsymbol{y}) \leq \lambda,\ d(\boldsymbol{y}, \ddot{\boldsymbol{\xi}}) \leq r\}$$

with r sufficiently small. Here Λ_s is a Lipshitz parameter defined by

$$\Lambda_s(x) := \|\mathrm{Lip}_s(x, \cdot)\|, \qquad \mathrm{Lip}_s(y, \tau) := \sup|y(\sigma) - y(\sigma')|/|\sigma - \sigma'|,$$

where the supremum is taken over all σ, σ' with $\sigma \neq \sigma'$ ranging over $[\tau - s, \infty)$ for $\tau > 0$ and $(-\infty, \tau + s]$ for $\tau < 0$.

4. Sketch of proof

4.1. Existence

We need to show that, for small enough ε, there exists a $K < 1$ such that

$$d(\boldsymbol{\Gamma}(\boldsymbol{y}^{(1)}), \boldsymbol{\Gamma}(\boldsymbol{y}^{(2)})) < K d(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \tag{8}$$

for all $\boldsymbol{y}^{(1)}$, $\boldsymbol{y}^{(2)}$ in $X_{s,\lambda,r}$, and that $\boldsymbol{\Gamma}$ maps this set into itself. In the following, if q is any quantity defined in terms of the $\boldsymbol{y}$, then we set $q^{(A)} := q(\boldsymbol{y}^{(A)})$ $(A = 1, 2)$, $\delta q := q^{(1)} - q^{(s)}$.

The condition $d(\boldsymbol{y}, \ddot{\boldsymbol{\xi}}) \leq r$ implies that $|\dot{y}_A^{(\tau)} - \dot{\xi}_A^{(\tau)}| < \nu$ for some ν tending to 0 with r. Thus for small enough r we have

$$|\mathrm{d}x_A^a/\mathrm{d}t| \equiv |\dot{x}^a/\dot{x}^0| < V_1$$

for some $V_1 < 1$ $(a = 1, 2, 3)$ and

$$|\dot{x}^{\mu}_{A}| < V_2.$$

(Note that τ is not necessarily proper time, and $\dot{x}^{\mu}$ cannot *a priori* be assumed time-like: see remark 3, § 2.)

From this a simple geometrical argument gives

$$R_{12}(\tau) \geqslant \begin{cases} C_1\tau - C_2 & (\tau < 0) \\ C_3\tau - C_2 & (\tau > 0) \end{cases}$$

for $C_1 > 1$, $0 < C_3 < 1$ and $C_2 > 0$, depending on V_i and the initial conditions. By iteration we obtain relations of the form

$$_kR_1(\tau) := R_{12} \circ R_{21} \ldots (\tau) \geqslant {_kA_{\pm}}\tau - B \qquad (\tau \gtrless 0) \tag{9}$$

for k retardations. Also from geometry, one can show that

$$(\delta_k R_1)(\tau) \leqslant C_4 \sup_{\tau'}(\delta x_A(\tau')) \tag{10}$$

where τ' ranges over an interval $[\alpha\tau - \gamma, \beta\tau + \gamma]$ for constants α, β, γ depending on the V_i and initial conditions. The proof of this is again by first proving a relation with $k = 1$ and then iterating. This is the only point at which the multiplicity of the iterations comes in.

Now let $f^{(A)}: \mathbb{R} \to \mathbb{R}^4$ $(A = 1, 2)$ be any two functions. It is clear that

$$\begin{aligned} &|\delta(f \circ {_kR_A})(\tau)| \\ &\quad \leqslant |f^{(1)}({_kR^{(1)}_A}(\tau)) - f^{(2)}({_kR^{(1)}_A}(\tau))| + |f^{(2)}({_kR^{(1)}_A}(\tau)) - f^{(2)}({_kR^{(2)}_A}(\tau))| \\ &\quad \leqslant |(\delta f)({_kR^{(1)}_A}(\tau))| + (\delta_k R_A)(\tau)\ \mathrm{Lip}_s(f^{(2)}, \alpha'\tau) \end{aligned} \tag{11}$$

for constants s, α' depending on α, β, γ. Moreover, the definition of the norm (7) gives

$$|\delta\ddot{x}(\tau)| < \begin{cases} C_5(1+\tau)^{-2} \\ C_6(1-\tau)^{-2-\alpha}, \end{cases} \qquad |\delta\dot{x}(\tau)| < \begin{cases} C_7 + C_8(1+\tau)^{-1} \\ C_9(1-\tau)^{-1-\alpha} \end{cases},$$

$$|\delta x(\tau)| < \begin{cases} C_{10}\tau + C_{11} \\ C_{12}(1-\tau)^{\alpha}, \end{cases} \qquad \begin{cases} \tau > 0 \\ \tau < 0 \end{cases}.$$

We now use these expressions and (10) to evaluate the result of taking f to be x_A, $\dot{x}_A$ and $\ddot{x}_A$ in turn in (11). This gives

$$|\overset{(0)}{\delta x}_{Ar}(\tau)| \leqslant \|\delta x\| \begin{cases} C_{13}(1+\tau) \\ C_{14}(1-\tau)^{-\alpha} \end{cases} \qquad \begin{cases} \tau > 0 \\ \tau < 0, \end{cases}$$

$$|\overset{(1)}{\delta x}_{Ar}(\tau)| \leqslant \|\delta x\| \begin{cases} C_{15}(1+[1+\tau]^{-1}) \\ C_{16}(1-\tau)^{-1-\alpha} \end{cases} \qquad \begin{cases} \tau > 0 \\ \tau < 0, \end{cases}$$

$$|\overset{(2)}{\delta x}_{Ar}(\tau)| \leqslant \|\delta x\| \begin{cases} C_{17}(1+\tau)^{-1} \\ C_{18}(1-\tau)^{-2-\alpha} \end{cases} \qquad \begin{cases} \tau > 0 \\ \tau > 0 \end{cases}.$$

Note that C_{17} and C_{18} depend on λ through the Lip_s in (11) (while for the lower derivatives, x and $\dot{x}$, the Lipshitz constant is given by the sup of $\dot{x}$ and $\ddot{x}$ respectively). This is why we need a restriction on the Lipshitz constant of $\ddot{x}$, through λ.

We now insert these variations into the function f of (1), estimating the result by condition (4b). We get

$$|\delta f_A(\overset{(0)}{x}_{A0}(\tau), \ldots)| \leqslant \|\delta \boldsymbol{x}\| t_A \begin{cases} C_{19}(1+\tau)^{-2} & \tau > 0 \\ C_{20}(1-\tau)^{-3-\alpha} & \tau < 0 \end{cases} \tag{12}$$

where (9) has been used repeatedly to relate estimates in terms of ${}_kR(\tau)$ to those in terms of τ.

Finally we insert this in the integral (6) to verify (8). To check that $\boldsymbol{\Gamma}$ maps $X_{s,\lambda,r}$ into itself we need to examine the λ and r conditions on $\boldsymbol{\Gamma}(\boldsymbol{x})$. For λ, the Lipshitz condition, write (6) (with $\boldsymbol{y} = \ddot{\boldsymbol{x}}$) as

$$y_A(\sigma) - y_A(\sigma') = \int_{\sigma'}^{\infty} [\exp(\sigma - \sigma') - 1] \exp[(\sigma' - \tau')/t_A] f_A \, \mathrm{d}\tau' + \int_{\sigma}^{\sigma'} \exp[(\sigma - \tau')/t_A] f_A \, \mathrm{d}\tau'.$$

Then

$$|y_A(\sigma) - y_A(\sigma')| \leqslant \tfrac{1}{2}|\sigma - \sigma'||y_A(\sigma')| + |\sigma - \sigma'| \sup_{[\sigma, \sigma']} |f(\tau')|$$

(for $|\sigma - \sigma'| < \frac{1}{2}$); whence the required Lipshitz condition follows from the relations derived earlier.

For the r condition, note that $\boldsymbol{\xi}$ has been chosen so that $\|\ddot{\boldsymbol{\xi}} - \Gamma(\xi)\| \to 0$ as $\varepsilon \to 0$ (i.e. $\boldsymbol{\xi}$ is 'approximately' a fixed point of Γ already). Then the contraction property (8) shows that, for small enough ε, $X_{s,\infty,r}$ is mapped into $X_{s,\lambda,r}$.

4.2. *Uniqueness*

Any fixed point of $\boldsymbol{\Gamma}$ must be unique in the domain where $\boldsymbol{\Gamma}$ is contractive. So we simply have to show that any fixed point satisfying the boundary conditions is in fact in $X_{s,\lambda,r}$. Since we have just noted that $\boldsymbol{\Gamma}$ maps $X_{s,\infty,r}$ to $X_{s,\lambda,r}$ for small enough ε and r, we need only verify that $d(y, \ddot{\boldsymbol{\xi}}) < r$ for solutions satisfying the conditions, i.e. that the conditions imply that

$$|\ddot{x}_A(\tau) - \ddot{\xi}_A(\tau)| < r(1+\tau)^{-2}. \tag{13}$$

Let τ_0 ($\leqslant \infty$) be the largest value such that

$$|\dot{x}_A(\tau) - \dot{\xi}_A(\tau)| \leqslant \tfrac{1}{2}|\dot{\xi}_A(\tau)| \qquad \text{for all } \tau < \tau_0. \tag{14}$$

Such a τ_0 exists by virtue of condition (iii). If $\tau_0 < \infty$, then we must have

$$|\dot{x}_A(\tau_0) - \dot{\xi}_A(\tau_0)| = \tfrac{1}{2}|\dot{\xi}_A(\tau_0)|$$

for either $A = 1$ or $A = 2$. But for a fixed point it is easily seen that, for small enough ε, this is impossible by virtue of (4a). Thus (14) holds with $\tau_0 = \infty$.

A further application of (4a) now gives $|\ddot{x}_A(\tau) - \ddot{\xi}_A(\tau)| <$ constant $(1+\tau)^{-1-\gamma}$; and another repetition yields (13).

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